1. (a) A RANDOM VARIABLE ON A SAMPLE SPACE Ω IS SIMPLY A FUNCTION $X: \Omega \to \mathbb{R}$ THAT ASSIGNS A REAL NUMBER $X(s)$ TO EACH SAMPLE $s \in \Omega$.

(b) THE INDICATOR VARIABLE 1_A FOR AN EVENT $A \subset \Omega$ ASSIGNS 1 TO EACH SAMPLE $s \in A$ AND 0 TO ALL OTHERS: $1_A(s) = \begin{cases} 1 & \text{IF } x \in A \\ 0 & \text{IF } x \notin A \end{cases}$

THESE ARE USEFUL BECAUSE THEY ALLOW US TO APPLY RANDOM-VARIABLE REASONING TO EVENTS (SUCH AS EXPECTED VALUE, BELOW!).

(c) IF X, Y ARE RANDOM VARIABLES ON Ω AND $c \in \mathbb{R}$, WE CAN COMBINE THESE IN THE SAME WAY THAT WE COMBINE REGULAR VARIABLES:

(i) $cX: \Omega \to \mathbb{R}$ IS DEFINED BY $s \xmapsto{cX} c \cdot X(s)$ (SCALE X'S VALUES BY c)

(ii) $X+Y: \Omega \to \mathbb{R}$ IS DEFINED BY $s \xmapsto{X+Y} X(s) + Y(s)$ (ADD THE VALUES OF X & Y)

(iii) $\min(X,Y): \Omega \to \mathbb{R}$ IS DEFINED BY $s \mapsto \min(X(s), Y(s))$ (ASSIGN THE SMALLER VALUE)

(iv) $\max(X,Y): \Omega \to \mathbb{R}$ IS DEFINED BY $s \mapsto \max(X(s), Y(s))$ (ASSIGN THE LARGER VALUE)

(d) FOR A GIVEN VALUE $x \in \mathbb{R}$, WE DEFINE $\mathbb{P}(X=x) = \mathbb{P}(\{s \in \Omega : X(s) = x\})$, I.E., THE SUM OF THE PROBABILITIES OF ALL SAMPLES s FOR WHICH $X(s) = x$; THIS DEFINES A PROBABILITY DISTRIBUTION ON THE RANGE (I.E., SET OF ALL OUTPUT VALUES) OF X.

2. IF X IS A RANDOM VARIABLE ON $(\Omega, \mathbb{P})$, WE DEFINE ITS EXPECTED VALUE BY

$$\boxed{E(X) = \sum_{s \in \Omega} \underbrace{X(s)}_{\text{VALUES}} \cdot \overbrace{\mathbb{P}(s)}^{\text{WEIGHTS}}}$$

$E(X)$ GIVES A "WEIGHTED AVERAGE" OF THE VALUES OF THE RANDOM VARIABLE X OVER ALL SAMPLES s, WITH THE WEIGHTS GIVEN BY $\mathbb{P}$.

ESSENTIALLY, $E(X)$ TELLS US THE AVERAGE RESULT WE'D GET FROM THE RANDOM VARIABLE X IF WE SAMPLE IT A BUNCH OF TIMES.

THE MOST IMPORTANT PROPERTY OF EXPECTED VALUE IS THAT IT IS LINEAR:

- $E(c \cdot X) = c \cdot E(X)$

AND
- $E(X+Y) = E(X) + E(Y)$

FOR ANY RANDOM VARIABLES X, Y ON Ω AND ANY CONSTANT $c \in \mathbb{R}$.

3. Rolling two fair six-sided dice, we can simply chart the 36 equally-likely possibilities, given the value X of the first die & Y of the second:

(a)(i) $X+Y$:

X (columns), Y (rows)

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

a	2	3	4	5	6	7	8	9	10	11	12
$\mathbb{P}(X+Y=a)$	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

Expected value: $\frac{1}{36}\cdot 2+\frac{2}{36}\cdot 3+\dots+\frac{1}{36}\cdot 12=\frac{252}{36}=\underline{7}$

(ii) $X-Y$:

X (columns), Y (rows)

	1	2	3	4	5	6
1	0	1	2	3	4	5
2	-1	0	1	2	3	4
3	-2	-1	0	1	2	3
4	-3	-2	-1	0	1	2
5	-4	-3	-2	-1	0	1
6	-5	-4	-3	-2	-1	0

a	-5	-4	-3	-2	-1	0	1	2	3	4	5
$\mathbb{P}(X-Y=a)$	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

Expected value: $\frac{1}{36}(-5)+\frac{2}{36}(-4)+\dots+\frac{1}{36}(5)=\underline{0}$

(iii) $\min(X,Y)$:

X (columns), Y (rows)

	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	3	3	3	3
4	1	2	3	4	4	4
5	1	2	3	4	5	5
6	1	2	3	4	5	6

a	1	2	3	4	5	6
$\mathbb{P}(\min(X,Y)=a)$	$11/36$	$9/36$	$7/36$	$5/36$	$3/36$	$1/36$

Expected value: $\frac{11}{36}\cdot 1+\frac{9}{36}\cdot 2+\dots+\frac{1}{36}\cdot 6=\underline{\frac{91}{36}}$

(iv) $\max(X,Y)$:

X (columns), Y (rows)

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

a	1	2	3	4	5	6
$\mathbb{P}(\max(X,Y)=a)$	$1/36$	$3/36$	$5/36$	$7/36$	$9/36$	$11/36$

Expected value: $\frac{1}{36}\cdot 1+\frac{3}{36}\cdot 2+\dots+\frac{11}{36}\cdot 6=\underline{\frac{161}{36}}$

(b) Adding values for $(X+Y)$ and $(X-Y)$ gives:

X (columns), Y (rows)

	1	2	3	4	5	6
1	2	4	6	8	10	12
2	2	4	6	8	10	12
3	2	4	6	8	10	12
4	2	4	6	8	10	12
5	2	4	6	8	10	12
6	2	4	6	8	10	12

a	2	4	6	8	10	12
$\mathbb{P}(\dots=a)$	$6/36$	$6/36$	$6/36$	$6/36$	$6/36$	$6/36$

This is just $2X$ — the Y & $-Y$ cancel at each $s\in\Omega$.

(c) Adding values for $\min(X,Y)$ and $\max(X,Y)$ gives:

X (columns), Y (rows)

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

(See (a)(i)!)

Note that the strange numbers in (a)(iii, iv) add up to the nice expected value of $\frac{252}{36}=7$ of (a)(i)!

This is just $X+Y$ — at each $s\in\Omega$, one of $X(s), Y(s)$ is the minimum and the other is the maximum, so the sums are identical.

4. JUST LIKE #3, BUT WITH DIFFERENT NUMBERS:

X:

Y \ X	1	2	2	3	3	4
1	1	2	2	3	3	4
3	1	2	2	3	3	4
4	1	2	2	3	3	4
5	1	2	2	3	3	4
6	1	2	2	3	3	4
8	1	2	2	3	3	4

a	1	2	3	4
$\mathbb{P}(X=a)$	$6/36$	$12/36$	$12/36$	$6/36$

Y:

Y \ X	1	2	2	3	3	4
1	1	1	1	1	1	1
3	3	3	3	3	3	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5
6	6	6	6	6	6	6
8	8	8	8	8	8	8

a	1	3	4	5	6	8
$\mathbb{P}(Y=a)$	$6/36$	$6/36$	$6/36$	$6/36$	$6/36$	$6/36$

$X+Y$

Y \ X	1	2	2	3	3	4
1	2	3	3	4	4	5
3	4	5	5	6	6	7
4	5	6	6	7	7	8
5	6	7	7	8	8	9
6	7	8	8	9	9	10
8	9	10	10	11	11	12

a	2	3	4	5	6	7	8	9	10	11	12
$\mathbb{P}(X+Y=a)$	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

SAME DISTRIBUTION OF SUMS AS USUAL DICE!

5. $(\Omega, \mathbb{P})$: FAIR COIN FLIPPED 9 TIMES

(a) $\mathbb{P}(A_{123}) = \underbrace{2}_{\text{H OR T}} \cdot \dfrac{1^3 \cdot 2^6}{2^9}$ — $1^3 \cdot 2^6$: ONE CHOICE FOR FIRST 3 FLIPS, 2 FOR THE REST; 2^9: TWO CHOICES FOR ALL FLIPS IN Ω

$= \dfrac{1}{4}$

(b) $\mathbb{P}(A_{234}) = \frac{1}{4}$ (ANALYSIS AS FOR A_{123})

(c) $\mathbb{P}(A_{123} \mid A_{234}) = \dfrac{\mathbb{P}(\text{FIRST FOUR ROLLS ALL THE SAME})}{\mathbb{P}(A_{234})} = \dfrac{2 \cdot \frac{1^4 \cdot 2^5}{2^9}}{1/4} = \dfrac{1}{2}$

$\therefore$ THESE EVENTS ARE <u>NOT</u> INDEPENDENT: IF THEY WERE, THEN $\mathbb{P}(A_{123} \mid A_{234})$ WOULD EQUAL $\mathbb{P}(A_{123}) = \frac{1}{4}$.

(INTUITIVELY, IF THE SECOND, THIRD, AND FOURTH FLIPS ARE IDENTICAL, IT'S 50-50 FOR THE FIRST FLIP TO MATCH.)

(d) $E(\mathbb{1}_{A_{123}}) = \mathbb{P}(A_{123}) = \frac{1}{4}$, AND THE SAME FOR ALL OF THE OTHERS!

(e)
$$\begin{aligned} E(\mathbb{1}_{A_{123}} + \mathbb{1}_{A_{234}} + \cdots + \mathbb{1}_{A_{789}}) &= E(\mathbb{1}_{A_{123}}) + E(\mathbb{1}_{A_{234}}) + \cdots + E(\mathbb{1}_{A_{789}}) \\ &= \mathbb{P}(A_{123}) + \mathbb{P}(A_{234}) + \cdots + \mathbb{P}(A_{789}) \\ &= \frac{1}{4} + \frac{1}{4} + \cdots + \frac{1}{4} \\ &= 7 \cdot \frac{1}{4} = \boxed{\frac{7}{4}} \end{aligned}$$

$\therefore$ WE EXPECT $\frac{7}{4}$ (A LITTLE UNDER TWO) SEQUENCES OF 3 CONSECUTIVE IDENTICAL FLIPS, ON AVERAGE.

6. $(\Omega, \mathbb{P})$: $n!$ PERMUTATIONS OF $(1, 2, \ldots, n)$, WITH UNIFORM DISTRIBUTION.

(a) $\mathbb{P}(\text{FIRST \# IS } 1) = \dfrac{1 \cdot (n-1)!}{n!}$ → ONE CHOICE FOR FIRST #, THE REST PERMUTED IN $(n-1)!$ WAYS.

$= \dfrac{1}{n}$

$\mathbb{P}(\text{SECOND \# IS } 2) = \dfrac{1}{n}$ BY SIMILAR REASONING.

$\mathbb{P}(\text{FIRST \# IS } 1 \wedge \text{SECOND \# IS } 2) = \dfrac{1 \cdot 1 \cdot (n-2)!}{n!}$ → ONE CHOICE FOR EACH OF THE FIRST TWO, THE REST PERMUTED IN $(n-2)!$ WAYS.

$= \dfrac{1}{n(n-1)}$

THIS $\neq \frac{1}{n} \cdot \frac{1}{n}$, SO THESE EVENTS ARE <u>NOT</u> INDEPENDENT.

(INTUITIVELY, IF THE FIRST # IS 1, THEN THERE IS ONE FEWER # THAT COULD BE IN THE SECOND SLOT, ∴ SLIGHTLY HIGHER PROBABILITY OF A 2 THERE.)

(b) IN GENERAL, $\mathbb{P}(\underbrace{K^{TH} \text{ \# IS } K}_{A_K}) = \frac{1}{n}$, SO THE INDICATORS $\mathbb{1}_{A_K}$ HAVE $E(\mathbb{1}_{A_K}) = \mathbb{P}(A_K) = \frac{1}{n}$.

(c) $E(\mathbb{1}_{A_1} + \mathbb{1}_{A_2} + \cdots + \mathbb{1}_{A_n}) = E(\mathbb{1}_{A_1}) + E(\mathbb{1}_{A_2}) + \cdots + E(\mathbb{1}_{A_n})$

$= \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots + \mathbb{P}(A_n)$

$= \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} = \boxed{1}$

∴ WE EXPECT (ON AVERAGE) <u>ONE</u> # IN ITS CORRESPONDING SLOT.